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On Extremal Sets and Strong Unicity Constants for Certain C^∞ Functions

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For each f continuous on the interval I, let $B_n(f)$ denote the best uniform polynomial approximation of degree less than or equal to n. Let $M_n(f)$ denote the corresponding strong unicity constant. For a certain class of nonrational functions F, it is shown that there exist positive constants α and β and a natural number N such that $\alpha n \leqslant M_n(f) \leqslant \beta n$ for $n \geqslant N$. The results of the present paper also provide concise estimates to the location of the extreme points of $f - B_n(f)$. The set F includes the functions $f_\alpha(x) = e^{\alpha x}$, $\alpha \neq 0$.

1. INTRODUCTION

Let $C(I)$ denote the space of real valued, continuous functions on the interval $I = [-1, 1]$, and let $\Pi_n \subseteq C(I)$ be the space of polynomials of degree at most n. Denote the uniform norm on $C(I)$ by $\|\cdot\|$. For each $f \in C(I)$ with best approximation $B_n(f)$ from Π_n, there is a smallest constant $M_n(f) > 0$ such that, for any $p \in \Pi_n$,

$$\|p - B_n(f)\| \leqslant M_n(f)(\|f - p\| - \|f - B_n(f)\|). \tag{1.1}$$

Inequality (1.1) is the well-known strong unicity theorem [3, p. 80], and hereafter $M_n(f)$ is defined to be the strong unicity constant.

The behavior of the sequence

$$\{M_n(f)\}_{n=0}^{\infty}$$

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has been the subject of several recent papers. In addition to the references of the current paper the interested reader is directed to a recent survey by Bartelt and Schmidt [1] and the references of [5, 7].

In the present paper the authors consider the order of the growth of $M_n(f)$ as function of the dimension of the approximating space Π_n.

DEFINITION 1. Let $f \in C(I)$, and suppose there exist positive constants α and β, a natural number N, and a positive real-valued function c with domain the natural numbers satisfying

$$\alpha c(n) \leqslant M_n(f) \leqslant \beta c(n) \qquad \text{for all } n \geqslant N.$$

Then $M_n(f)$ is said to be of precise order $c(n)$.

In [5] it is shown that if $f(x) = 1/(x-\lambda)$, $\lambda \geqslant 2$, $x \in I$, then $M_n(f)$ is of precise order n. Henry and Swetits have subsequently established that $M_n(f)$ is of precise order n for every rational function f of a particular type [9].

A primary objective of the current paper is to find the precise order of $M_n(f)$ for $f(x) = e^x$ and for every function in a related class of non-rational functions.

Another main objective is to concisely estimate the extreme points of $e_n(f) = f - B_n(f)$ for the class of non-rational functions alluded to above.

Several residual observations relating to interpolation are also made.

2. PRELIMINARIES

For $f \in C(I)$, $e_n(f)(x) = f(x) - B_n(f)(x)$. Let

$$E_n(f) = \{x \in I : |e_n(f)(x)| = \|e_n(f)\|\}$$

be the extreme points of the error curve $e_n(f)$. Given $n+2$ distinct points $\{x_i\}_{i=0}^{n+1} \subseteq E_n(f)$, define $q_{in} \in \Pi_n$, $i = 0,..., n+1$, by

$$q_{in}(x_j) = \operatorname{sgn} e_n(f)(x_j), \qquad j = 0,..., n+1, j \neq i, i = 0,..., n+1. \tag{2.1}$$

Similarly define $Q_{n+1} \in \Pi_{n+1}$ by

$$Q_{n+1}(x_j) = \operatorname{sgn} e_n(f)(x_j), \qquad j = 0, 1,..., n+1. \tag{2.2}$$

If $E_n(f)$ consists of precisely $n+2$ points, then it is known [6] that

$$M_n(f) = \max_{0 \leqslant i \leqslant n+1} \|q_{in}\|. \tag{2.3}$$

The following four theorems are fundamental to the ensuing analysis.

THEOREM 1 (Rowland [14]). *Suppose for functions f and g that $f^{(n+2)}$ and $g^{(n+2)}$ are continuous on I and that $f^{(n+1)}$ and $g^{(n+1)}$ are positive on I. Let*

$$-1 \leqslant y_0(g) < y_1(g) < \cdots < y_{n+1}(g) \leqslant 1$$

and

$$-1 \leqslant y_0(f) < y_1(f) < \cdots < y_{n+1}(f) \leqslant 1$$

denote the extreme points of $e_n(g)$ and $e_n(f)$, respectively. If

$$\frac{g^{(n+2)}(x)}{g^{(n+1)}(x)} < \frac{f^{(n+2)}(x)}{f^{(n+1)}(x)} \qquad \text{for all } x \in I,$$

then

$$y_k(g) < y_k(f) \qquad \text{for } k = 1,..., n.$$

Theorem 1 is actually a special case of Theorem 3.1 and Corollary 3.2 in [14].

Now let

$$z_k = \cos \frac{(n+1-k)}{n+1} \pi, \qquad \zeta_k = \cos \frac{(n-k)}{n} \pi. \tag{2.4}$$

Then z_k, $k = 0,..., n+1$ and ζ_k, $k = 0,..., n$ are the extreme points of T_{n+1} and T_n, respectively, where T_j is the jth degree Chebyshev polynomial. Suppose in addition to the hypotheses of Theorem 1 it is assumed that $f^{(n+2)} > 0$. Then one can conclude that

$$z_k < y_k(f) < \zeta_k, \qquad 1 \leqslant k \leqslant n. \tag{2.5}$$

In fact, the first part of inequality (2.5) follows directly from Theorem 1 by choosing $g(x) = x^{n+1}$. The proof of the second part is given in [10, Theorem 81, p. 101].

THEOREM 2 (Bartelt and Schmidt [2]). *If $f \in C(I) - \Pi_n$, then*

$$M_n(f) = \max_{p \in \Pi_n} \{\|p\|: \operatorname{sgn} e_n(f)(x)\, p(x) \leqslant 1 \text{ for } x \in E_n(f)\}.$$

THEOREM 3 [7]. *Let $f \in C^{n+2}(I)$, with $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \neq 0$ for*

$x \in I$. If Q_{n+1} is determined by $E_n(f)$ as in (2.2), and if a_{n+1} is the coefficient of x^{n+1} in Q_{n+1}, then

(a) $$\|Q_{n+1}\| \leqslant M_n(f), \tag{2.6}$$

(b) $$2^n \leqslant |a_{n+1}|, \tag{2.7}$$

and

(c) $$2n+1 < M_n(f). \tag{2.8}$$

The next theorem will be utilized below to estimate the location of extreme points of the error curves for certain rational and non-rational functions.

THEOREM 4. *Suppose that $R > 0$ and $|S| > 1$ are fixed real numbers. Select N such that for all $n \geqslant N$, $R(n+2) - (|S|+1) > 0$ for all $x \in I$. Define the rational function r_n by*

$$r_n(x) = \frac{1}{R(n+2) + S - x}, \qquad x \in I.$$

Then there exist constants $\bar{R}$ and $\bar{S}$ not depending on n and a natural number $\bar{N}$ such that

$$\bar{R}n \leqslant M_n(r_n) \leqslant \bar{S}n, \qquad n \geqslant \bar{N}. \tag{2.9}$$

Proof. The set of extreme points $E_n(r_n)$ of $e_n(r_n)$ consists of precisely $n+2$ points. Label these points as

$$-1 = w_0^n < w_1^n < \cdots < w_n^n < w_{n+1}^n = 1.$$

If Q_{n+1}^n is as in (2.2), then

$$Q_{n+1}^n(w_i^n) = \operatorname{sgn} e_n(r_n)(w_i^n), \qquad i = 0, \ldots, n+1. \tag{2.10}$$

Let a_{n+1}^n be the coefficient of x^{n+1} in the Q_{n+1}^n defined by (2.10). Inequality (2.32) in [5] shows that $\|Q_{n+1}^n\| \leqslant An$, where A does not depend on n. Since (2.7) implies that $2^n \leqslant |a_{n+1}^n|$, an examination of the proof of Theorem 3 in [5] reveals that (2.9) will follow providing

$$|a_{n+1}^n| \leqslant \mu 2^n, \tag{2.11}$$

where μ is independent of n. The remainder theorem for classical Lagrange interpolation [3, p. 60] implies that

$$\frac{e_n(r_n)(x)}{\|e_n(r_n)\|} - q_{in}^n(x) = \frac{r_n^{(n+1)}(\xi)}{(n+1)!\,\|e_n(r_n)\|} \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (x - w_j^n), \tag{2.12}$$

where ξ is between x and an appropriate w_k^n, and q_{in}^n is defined by (2.1), $i = 0,..., n+1$. Evaluating (2.12) at w_i^n yields

$$\operatorname{sgn} e_n(r_n)(w_i^n) - q_{in}^n(w_i^n) = \frac{r_n^{(n+1)}(\xi_i)}{(n+1)!\, \|e_n(r_n)\|} \prod_{\substack{j=0\\ j\neq i}}^{n+1} (w_i^n - w_j^n).$$

Comparing q_{in}^n and Q_{n+1}^n (also see [5, Eq. (2.17)]) gives

$$q_{in}^n(x) = Q_{n+1}^n(x) - a_{n+1}^n \prod_{\substack{j=0\\ j\neq i}}^{n+1} (x - w_j^n). \tag{2.13}$$

Evaluating (2.13) at w_i^n and utilizing the above yields

$$a_{n+1}^n = \frac{r_n^{(n+1)}(\xi_i)}{(n+1)!\, \|e_n(r_n)\|}. \tag{2.14}$$

For any $f \in C^{(n+1)}(I)$, it is known that

$$\|e_n(f)\| = \frac{f^{(n+1)}(\xi)}{2^n(n+1)!}, \tag{2.15}$$

where $-1 < \xi < 1$ (see [10, p. 78]). Consequently (2.14) implies that

$$a_{n+1}^n = 2^n \frac{r_n^{(n+1)}(\xi_i)}{r_n^{(n+1)}(\eta_i)},$$

where $-1 \leqslant \xi_i,\ \eta_i \leqslant 1$. Therefore

$$a_{n+1}^n = 2^n \left[\frac{((S-\eta_i)/R(n+2)) + 1}{((S-\xi_i)/R(n+2)) + 1}\right]^{n+2}. \tag{2.16}$$

Equality (2.16) implies that

$$|a_{n+1}^n| \leqslant 2^n \left[\frac{1 + (|S|+1)/R(n+2)}{1 - (|S|+1)/R(n+2)}\right]^{n+2}.$$

Therefore there exists an N^* such that for $n \geqslant N^*$,

$$|a_{n+1}^n| \leqslant 2^{n+1} \exp\left(2\,\frac{|S|+1}{R}\right).$$

Consequently, for all $n \geqslant \bar{N} = \max(N, N^*)$, (2.11) is verified with $\mu = 2\exp(2((|S|+1)/R))$. ∎

The superscript notation of Theorem 4 was utilized to emphasize the dependence on dimension. Hereafter this dependence is assumed and consequently the superscripts are omitted.

In Section 3 we define **F**, a class of non-rational functions which includes the exponential function. In Theorem 6 of Section 3 we establish that $M_n(f)$ is of precise order n if $f \in \mathbf{F}$. The steps we take to prove this result are:

(A) identifying a rational function whose error function has extreme points sufficiently close to those of $e_n(f)$, and

(B) using Theorem 4 above.

The following theorem estimates the closeness of the extreme points of the error functions for a certain pair of rational functions and provides a bridge to establishing (A).

THEOREM 5. *Let $\alpha \geqslant \beta > 0$ be constants not depending on n. Define*

$$U_n(x) = \frac{1}{\alpha(n+2)+2-x} \qquad \text{for } x \in I$$

and (2.17)

$$V_n(x) = \frac{1}{\beta(n+2)-2-x} \qquad \text{for } x \in I$$

where n is sufficiently large so that the denominators of U_n and V_n do not vanish on I. Let

$$-1 = u_0 < u_1 < \cdots < u_n < u_{n+1} = 1$$

and

$$-1 = v_0 < v_1 < \cdots < v_n < v_{n+1} = 1$$

be the extreme points of $e_n(U_n)$ and $e_n(V_n)$, respectively. If z_i and ζ_i are given by (2.4), *then*

$$\text{(i)} \qquad z_i < u_i < v_i < \zeta_i, \qquad i = 1,\dots, n; \tag{2.18}$$

and

$$\text{(ii)} \qquad \left| \frac{u_i - v_i}{z_i - \zeta_{i-1}} \right| \leqslant \frac{\hat{A}}{n}, \qquad i = 1,\dots, n, \tag{2.19}$$

where $\hat{A}$ is independent of i and n.

Proof. Select N such that for all $n \geqslant N$, U_n and V_n are defined for $x \in I$. Then

$$\frac{U_n^{(n+2)}(x)}{U_n^{(n+1)}(x)} = \frac{n+2}{\alpha(n+2)+2-x} < \frac{1}{\alpha}. \tag{2.20}$$

Also

$$\frac{V_n^{(n+2)}(x)}{V_n^{(n+1)}(x)} = \frac{n+2}{\beta(n+2)-2-x} > \frac{1}{\beta}. \tag{2.21}$$

Thus (2.20) and (2.21) imply that

$$\frac{U_n^{(n+2)}}{U_n^{(n+1)}} < \frac{V_n^{(n+2)}}{V_n^{(n+1)}},$$

and (i) now follows from Theorem 1 and (2.5).

To prove (ii), first let $\lambda_L = \alpha(n+2)+2$ and $\lambda_R = \beta(n+2)-2$. As in [10, p. 36], if $\tau = 1/\lambda$ and $\xi_1(\tau),..., \xi_n(\tau)$ are the interior extreme points of the error curve for the rational function $1/(\lambda - x)$, then

$$\frac{d\xi_i(\tau)}{d\tau} = \frac{1-\xi_i^2(\tau)}{n\sqrt{1-\tau^2}(1-\tau\xi_i(\tau))+(1-\tau^2)}, \qquad i = 1,..., n. \tag{2.22}$$

Let $\tau_L = 1/\lambda_L$, $\tau_R = 1/\lambda_R$. Then

$$|v_i - u_i| = |\xi_i(\tau_R) - \xi_i(\tau_L)|, \qquad i = 1, 2,..., n.$$

Utilizing the mean value theorem and (2.22) yields

$$|v_i - u_i| = \frac{1-\xi_i^2(\hat{\tau})}{n\sqrt{1-\hat{\tau}^2}(1-\hat{\tau}\xi_i(\hat{\tau}))+(1-\hat{\tau}^2)} |\tau_R - \tau_L|. \tag{2.23}$$

where $\tau_L < \hat{\tau} < \tau_R$. From the definitions of τ_L and τ_R we obtain

$$\frac{1}{\alpha(n+2)+2} < \hat{\tau} < \frac{1}{\beta(n+2)-2}.$$

Hence (2.23) implies that there exists a positive constant μ not depending on i or n such that

$$|v_i - u_i| \leqslant \frac{1-\xi_i^2(\hat{\tau})}{\mu n} \left| \frac{(\alpha-\beta)(n+2)+4}{[\alpha(n+2)+2][\beta(n+2)-2]} \right|, \qquad i = 1, 2,..., n. \tag{2.24}$$

Therefore to establish (ii) it is sufficient to show that

$$\frac{[1-\xi_i^2(\hat{\tau})]}{|z_i-\zeta_{i-1}|} \leqslant \bar{A}n, \qquad i=1,2,\ldots,n. \tag{2.25}$$

where $\bar{A}$ is independent of i and n. We note from (2.22) that $\xi_i'(\tau)>0$ for $0<\tau\leqslant\frac{1}{2}$, and therefore $\xi_i(\tau_L)<\xi_i(\hat{\tau})<\xi_i(\tau_R)$, $i=1,\ldots,n$. That is, $u_i<\xi_i(\hat{\tau})<v_i$, $i=1,\ldots,n$.

Now let $i=1$. Then from part (i) of the current theorem, $\xi_1(\hat{\tau})<v_1<\zeta_1$, which in turn implies that $|\xi_1(\hat{\tau})|>|\zeta_1|$. Therefore

$$\begin{aligned}\frac{1-\xi_1^2(\hat{\tau})}{|z_1-\zeta_0|} &\leqslant \frac{1-\zeta_1^2}{|1-\cos(1/(n+1))\,\pi|}\\ &=\frac{\sin^2(\pi/n)}{2[\sin(1/2(n+1))\,\pi]^2}.\end{aligned} \tag{2.26}$$

Since the right-hand side of (2.26) is bounded independent of n, (2.25) is satisfied for $i=1$.

Next suppose that $2\leqslant i\leqslant n$. Applying (2.4) and the mean value theorem to the left side of (2.25) yields

$$\frac{1-\xi_i^2(\hat{\tau})}{|\sin\mu_i|\,|n+1-i|}(n)(n+1). \tag{2.27}$$

where

$$\frac{i-1}{n}\pi<\mu_i<\frac{i}{n+1}\pi. \tag{2.28}$$

If $2\leqslant i\leqslant n/4$, then the remarks below (2.23), part (i) of the current theorem, (2.27), and the observation that $\sin x$ is increasing on $[\pi/n, (n/(n+1))(\pi/4)]$ yield

$$\begin{aligned}\frac{1-\xi_i^2(\hat{\tau})}{|\sin\mu_i|\,|n+1-i|}n(n+1) &\leqslant \frac{1-\zeta_i^2}{|\sin((i-1)/n)\,\pi|\,|n+1-i|}n(n+1)\\ &=\frac{\sin^2(i/n)\,\pi}{|\sin((i-1)/n)\,\pi|\,|n+1-i|}n(n+1)\\ &\leqslant\frac{2n(n+1)}{n+1-i}, \qquad i=2,\ldots,\left[\frac{n}{4}\right].\end{aligned} \tag{2.29}$$

Now assume that $n/4\leqslant i\leqslant 3n/4$. Then from (2.28),

$$\frac{[n/4]-1}{n}\pi\leqslant\mu_i\leqslant\frac{[3n/4]}{n+1}\pi,$$

and consequently for n sufficiently large

$$\frac{1-\xi_i^2(\hat{t})}{|\sin\mu_i|\,|n+1-i|}\,n(n+1)\leqslant 2\,\frac{(n)(n+1)}{(n+1-i)}. \tag{2.30}$$

Finally, the analysis needed to achieve (2.25) for $3n/4\leqslant i\leqslant n$ closely parallels that given above (2.29). Combining this observation with (2.26), (2.29), and (2.30) establishes (2.25) for $i=1,2,\dots,n$. Inequalities (2.24) and (2.25) imply conclusion (ii). ∎

Remark. Inequality (2.24) implies for the rational functions U_n and V_n that $\max_{1\leqslant i\leqslant n}|u_i-v_i|=O(1/n^2)$. For $\alpha=\beta$ in (2.17), (2.24) implies that

$$\max_{1\leqslant i\leqslant n}|u_i-v_i|=O\left(\frac{1}{n^3}\right). \tag{2.31}$$

Thus the maximum distance between corresponding extreme points of $e_n(U_n)$ and $e_n(V_n)$ is $O(1/n^2)$. This distance is to be contrasted with the maximum distance between corresponding extreme points of T_n and T_{n+1}, which is $O(1/n)$. Equation (2.31) demonstrates an even more striking comparison for $\alpha=\beta$. Utilization of (2.31) will provided concise estimates to the location of the extreme points of $e_n(f)$, where $f(x)=e^x$.

3. A Class of Non-Rational Functions

In this section we show that $M_n(f)$ is of precise order n for $f\in\mathbf{F}$, a class of non-rational functions which includes the exponential function $f(x)=e^x$.

Definition 2. Let $\mathbf{F}$ be the set of all $f\in C^\infty(I)$ satisfying

(a) $$f^{(n+1)}(x)\neq 0 \quad \text{on } I$$

and

(b) $$\frac{1}{\alpha}\leqslant\left|\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\right|\leqslant\frac{1}{\beta} \quad \text{on } I. \tag{3.1}$$

for all n sufficiently large, where $\alpha\geqslant\beta>0$ are constants depending on f but not on n.

Theorem 6. *Let $f\in\mathbf{F}$. Then $M_n(f)$ is of precise order n.*

Prior to effecting the proof of Theorem 6, necessary terminology is introduced, and three lemmas that facilitate the establishing of Theorem 6 are proven.

For any $f \in \mathbf{F}$, let $E_n(f) = \{x_i\}_{i=0}^{n+1}$, where

$$-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1. \tag{3.2}$$

For the extremal set (3.2), q_{in}, $i = 0,..., n+1$, and Q_{n+1} are defined as in (2.1) and (2.2), respectively.

LEMMA 1. *Let $f \in \mathbf{F}$. Then there exists a $\hat{\beta}$ not depending on n such that*

$$\left| \frac{f^{(n+1)}(\xi)}{f^{(n+1)}(\eta)} \right| \leqslant \hat{\beta} \tag{3.3}$$

for every ξ, $\eta \in I$.

Proof. Without loss of generality assume that $f^{(n+2)}(x) > 0$ on I (otherwise replace f by $-f$). By first assuming that $f^{(n+1)}(x) > 0$ on I we can show that

$$\frac{f^{(n+1)}(\xi)}{f^{(n+1)}(\eta)} \leqslant \frac{f^{(n+1)}(1)}{f^{(n+1)}(-1)} \leqslant e^{2/\beta}$$

for every ξ, $\eta \in I$. Similarly if we assume that $f^{(n+1)}(x) < 0$ on I, then we can show that

$$\frac{f^{(n+1)}(\xi)}{f^{(n+1)}(\eta)} \leqslant \frac{f^{(n+1)}(-1)}{f^{(n+1)}(1)} \leqslant e^{2/\alpha}.$$

By selecting $\hat{\beta} = e^{2/\beta}$ we have established (3.3).

LEMMA 2. *Let U_n and V_n be the rational functions defined in Theorem 5. If $f \in \mathbf{F}$ with extremal set (3.2) and if $f^{(n+1)}(x) f^{(n+2)}(x) > 0$, then*

$$z_i < u_i < x_i < v_i < \zeta_i, \qquad i = 1,..., n. \tag{3.4}$$

Proof. Without loss of generality, assume $f^{(n+2)}(x) > 0$ for $x \in I$. Then by (3.1)

$$\frac{1}{\alpha} \leqslant \frac{f^{(n+2)}(x)}{f^{(n+1)}(x)} \leqslant \frac{1}{\beta}.$$

Thus (2.20) and (2.21) imply that

$$\frac{U_n^{(n+2)}(x)}{U_n^{(n+1)}(x)} < \frac{f^{(n+2)}(x)}{f^{(n+1)}(x)} < \frac{V_n^{(n+2)}(x)}{V_n^{(n+1)}(x)}.$$

Theorem 1 and (2.18) now imply (3.4). ∎

LEMMA 3. *Let U_n, V_n, u_i, v_i, f, and x_i, $i=0,2,\dots,n+1$, be as in Lemma 2. Then there exists constants A and B not depending on i or n and a natural number N such that*

$$|u_i - x_i| < \frac{A}{n}|x_i - x_j|, \qquad j \neq i, \quad i = 0, 1,\dots, n+1, \tag{3.5}$$

$$|u_i - x_i| \leqslant 2\frac{A}{n}|x_j - u_i|, \qquad i = 0,\dots, n+1, \quad j \neq i, \tag{3.6}$$

and

$$|u_i - x_j| \leqslant \left[1 + \frac{B}{n}\right]|x_i - u_j|, \qquad i = 0,\dots, n+1, \quad i \neq j, \tag{3.7}$$

for all $n \geqslant N$.

Proof. We first establish (3.5). If $i=0$ or $n+1$, (3.5) is immediate. Suppose $1 \leqslant i \leqslant n$. By (3.2) and (2.18), it suffices to show (3.5) for $j = i-1$ and $j = i+1$. Suppose $j = i-1$. Then by (3.4)

$$\begin{aligned} x_i - x_{i-1} &= (\zeta_{i-1} - x_{i-1}) + (z_i - \zeta_{i-1}) + (u_i - z_i) + (x_i - u_i) \\ &\geqslant z_i - \zeta_{i-1} \\ &\geqslant (x_i - u_i)\left[\frac{z_i - \zeta_{i-1}}{v_i - u_i}\right]. \end{aligned}$$

Therefore

$$\frac{v_i - u_i}{z_i - \zeta_{i-1}} \geqslant \frac{x_i - u_i}{x_i - x_{i-1}} \geqslant 0, \qquad i = 1,\dots, n.$$

Now this inequality and (2.19) imply that

$$\frac{x_i - u_i}{x_i - x_{i-1}} \leqslant \frac{\hat{A}}{n}, \qquad i = 1,\dots, n;$$

thus for $j = i-1$, (3.5) is proven.

Now suppose $j = i+1$, where $1 \leqslant i \leqslant n-1$. Then

$$\begin{aligned} x_{i+1} - x_i = v_i - x_i + \zeta_i - v_i + z_{i+1} - \zeta_i + u_{i+1} - z_{i+1} \\ + x_{i+1} - u_{i+1}, \qquad i = 1, 2,\dots, n-1. \end{aligned}$$

Therefore

$$\begin{aligned} x_{i+1} - x_i &\geqslant (z_{i+1} - \zeta_i) \\ &\geqslant \frac{x_i - u_i}{v_i - u_i} \cdot \frac{z_i - \zeta_{i-1}}{z_i - \zeta_{i-1}} \cdot (z_{i+1} - \zeta_i). \end{aligned}$$

Hence

$$\frac{z_i-\zeta_{i-1}}{z_{i+1}-\zeta_i}\cdot\frac{v_i-u_i}{z_i-\zeta_{i-1}}\geqslant\frac{x_i-u_i}{x_{i+1}-x_i}. \tag{3.8}$$

But $(z_i-\zeta_{i-1})/(z_{i+1}-\zeta_i)$ is bounded independently of n for $i=1,\dots,n-1$. Now (3.8) and (2.19) imply for $j=i+1$, $i=1,\dots,n-1$, that

$$\frac{x_i-u_i}{x_{i+1}-x_i}\leqslant\frac{A_1}{n}, \tag{3.9}$$

where A_1 is independent of i and n. The proof of (3.5) will be complete if

$$\frac{x_n-u_n}{x_{n+1}-x_n}\leqslant\frac{A_2}{n}, \tag{3.10}$$

where A_2 is independent of n.

Note that since x_n and x_{n+1} are not separated by any extreme point of either T_n or T_{n+1}, a different argument than that given to establish (3.9) is required. For n sufficiently large

$$\begin{aligned}\frac{x_n-u_n}{x_{n+1}-x_n}&\leqslant\frac{v_n-u_n}{1-v_n}\\&\leqslant\frac{2(v_n-u_n)}{1-v_n^2}.\end{aligned} \tag{3.11}$$

By employing (2.24) and the observations below (2.25), (3.11) implies that

$$\begin{aligned}\frac{x_n-u_n}{x_{n+1}-x_n}&=\frac{1-\xi_n^2(\hat{\tau})}{1-v_n^2}\cdot\frac{B_1}{n^2}\\&\leqslant\frac{1-z_n^2}{1-v_n^2}\cdot\frac{B_1}{n^2}\\&\leqslant\frac{1}{1-v_n^2}\cdot\frac{B_2}{n^4},\end{aligned} \tag{3.12}$$

where B_1 and B_2 are independent of n. Let

$$h_n(x)=n(\lambda_n^2-1)^{1/2}\,T_n(x)+(\lambda_n x-1)\,T_n'(x),$$

where $\lambda_n=\beta(n+2)-2$. Then

$$h_n(v_i)=0, \tag{3.13}$$

$i = 1, \ldots, n$ (see [10, p. 35]). Also

$$\|h_n'\| \leqslant B_3 n^5, \tag{3.14}$$

where B_3 does not depend on n. On the other hand, (3.13) and Eq. (2.25) in [5] $(a = \lambda_n)$ show that

$$(1 - v_n^2)\, h_n'(v_n) \geqslant \bar{a} n^2, \tag{3.15}$$

where $\bar{a} > 0$ does not depend on n.

From (3.12),

$$\frac{x_n - u_n}{x_{n+1} - x_n} \leqslant \frac{h_n'(v_n)}{(1 - v_n^2)\, h_n'(v_n)} \cdot \frac{B_2}{n^4}.$$

Utilizing (3.14) and (3.15) in this equality establishes (3.10), where $A_2 = B_2 \cdot B_3/\bar{a}$. The proof of (3.5) is completed by selecting $A = \max(\hat{A}, A_1, A_2)$.

To prove (3.6) observe that by (3.5)

$$\begin{aligned} |u_i - x_i| &\leqslant \frac{A}{n} |x_i - x_k|, \qquad k \neq i, \\ &< \frac{A}{n} (|u_i - x_i| + |x_k - u_i|). \end{aligned}$$

This inequality implies that

$$\left(1 - \frac{A}{n}\right) |u_i - x_i| \leqslant \frac{A}{n} |x_k - u_i|, \qquad k \neq i.$$

Thus (3.6) follows for n sufficiently large.

We now prove (3.7). From (3.5) and (3.6) we have that

$$\begin{aligned} |u_i - x_j| &\leqslant |u_i - x_i| + |x_i - x_j| \\ &\leqslant \left(1 + \frac{A}{n}\right) (|x_i - u_j| + |u_j - x_j|) \\ &\leqslant \left(1 + \frac{A}{n}\right) \left[\left(1 + \frac{2A}{n}\right) |x_i - u_j| \right] \\ &\leqslant \left(1 + \frac{B}{n}\right) |x_i - u_j| \end{aligned}$$

for n sufficiently large, where B is independent of i and n. ∎

The above lemmas are now utilized to prove Theorem 6.

Proof of Theorem 6. Let $f \in \mathbf{F}$. Without loss of generality we may assume that $f^{(n+2)}(x) > 0$ for $x \in I$. For this part of the proof we also assume that $f^{(n+1)}(x) > 0$ on I.

Let U_n be the rational function defined in (2.17). Then (2.9) is valid for n sufficiently large. Let $q_{in} \in \Pi_n$, $i = 0, 1, \ldots, n+1$, be the polynomial satisfying (2.1) for U_n. Then by (2.3)

$$M_n(U_n) \doteq \max_{0 \leqslant i \leqslant n+1} \|q_{in}\|. \tag{3.16}$$

As in (2.13),

$$q_{in}(x) = Q_{n+1}(x) - a_{n+1} \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (x - u_j), \tag{3.17}$$

where Q_{n+1} is defined in (2.10) for U_n. Equation (3.17) implies

$$\|q_{in}\| + \|Q_{n+1}\| \geqslant |a_{n+1}| \max_{-1 \leqslant x \leqslant 1} \left| \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (x - u_j) \right|.$$

Therefore (2.6), (2.7), (2.9), and (3.16) yield

$$\max_{0 \leqslant i \leqslant n+1} \max_{-1 \leqslant x \leqslant 1} 2^n \prod_{\substack{j=0 \\ j \neq i}}^{n+1} |x - u_j| = O(n). \tag{3.18}$$

Let the extreme points for $e_n(f)$ be given by (3.2) and let P_{ln} satisfy

$$P_{ln}(x_i) = \operatorname{sgn} e_n(f)(x_i), \qquad i \neq l, \tag{3.19}$$

and (again using (2.3))

$$M_n(f) = \|P_{ln}\|.$$

By (3.19) and the classical remainder theorem for Lagrange interpolation,

$$P_{ln}(x) = \frac{e_n(f)(x)}{\|e_n(f)\|} - \frac{f^{(n+1)}_{(\xi)}}{\|e_n(f)\|\,(n+1)!} \prod_{\substack{j=0 \\ j \neq l}}^{n+1} (x - x_j),$$

where $-1 \leqslant \xi \leqslant 1$. From (2.15) this may be written as

$$P_{ln}(x) = \frac{e_n(f)(x)}{\|e_n(f)\|} - \frac{f^{(n+1)}(\xi)}{f^{(n+1)}(\eta)} 2^n \prod_{\substack{j=0 \\ j \neq l}}^{n+1} (x - x_j),$$

where $-1 \leqslant \eta \leqslant 1$. Lemma 1 now implies that

$$|P_{ln}(x)| \leqslant \hat{\beta}\, 2^n \prod_{\substack{j=0 \\ j \neq l}}^{n+1} |x - x_j| + 1.$$

Thus

$$|P_{ln}(u_k)| \leqslant \hat{\beta}\, 2^n \prod_{\substack{j=0 \\ j \neq l}}^{n+1} |u_k - x_j| + 1, \tag{3.20}$$

where $k = 0, 1,..., n+1$, $k \neq l$.

But Lemma 3 now implies for n sufficiently large that

$$\begin{aligned}
2^n \prod_{\substack{j=0 \\ j \neq l}}^{n+1} |u_k - x_j| \\
&= |u_k - x_k|\, 2^n \prod_{\substack{j=0 \\ j \neq k,l}}^{n+1} |u_k - x_j| \\
&\leqslant |u_k - x_k| \left(1 + \frac{B}{n}\right)^n \prod_{\substack{j=0 \\ j \neq k,l}}^{n+1} |x_k - u_j|\, 2^n \\
&\leqslant \frac{A}{n} \left(1 + \frac{B}{n}\right)^n |x_k - x_l| \prod_{\substack{j=0 \\ j \neq k,l}}^{n+1} |x_k - u_j|\, 2^n \\
&\leqslant \frac{A}{n} \left(1 + \frac{B}{n}\right)^n (|x_k - u_l| + |u_l - x_l|) \cdot \prod_{\substack{j=0 \\ j \neq k,l}}^{n+1} |x_k - u_j|\, 2^n \\
&\leqslant 2^n \frac{A}{n} \left(1 + \frac{B}{n}\right)^n \left[1 + \frac{2A}{n}\right] |x_k - u_l| \prod_{\substack{j=0 \\ j \neq k,l}}^{n+1} |x_k - u_j| \\
&\leqslant \frac{2A}{n} e^B \prod_{\substack{j=0 \\ j \neq k}}^{n+1} |x_k - u_j|\, 2^n.
\end{aligned}$$

This inequality and (3.18) now establish that

$$2^n \prod_{\substack{j=0 \\ j \neq l}}^{n+1} |u_k - x_j| \tag{3.21}$$

is bounded.

Now (3.20) and (3.21) imply that

$$|P_{ln}(u_k)| \leqslant \alpha, \qquad k = 0, 1, \ldots, n+1, \quad k \neq l.$$

where α does not depend on n. Therefore we may without loss of generality assume that

$$\operatorname{sgn} e_n(U_n)(x)\, P_{ln}(x)/\alpha \leqslant 1, \qquad x \in E_n(U_n)$$

(otherwise replace P_{ln} by $-P_{ln}$). Consequently Theorem 2 implies that $\|P_{ln}\| \leqslant \alpha M_n(U_n)$. That is, $M_n(f) \leqslant \alpha M_n(U_n) = O(n)$. This inequality and (2.8) now imply for $f^{(n+1)}(x) > 0$ on I that $M_n(f)$ is of precise order n.

Next suppose $f^{(n+1)}(x) < 0$ on I. Define $g_n(x)$ by $g_n(x) = (-1)^{n+2} f(-x)$. Since for any function $h \in C(I)$, $M_n(\alpha h) = M_n(h)$, $M_n(g_n) = M_n[(-1)^{n+2} g_n]$. Clearly $g_n^{(n+2)}(x) = f^{(n+2)}(-x) > 0$, and $g_n^{(n+1)}(x) = -f^{(n+1)}(-x) > 0$. Therefore the proof of the first part of Theorem 6 establishes that $M_n(g_n)$ is of precise order n, and hence $M_n[(-1)^{n+2} g_n]$ is of precise order n. Define h by $h(x) = f(-x)$, $x \in I$. Let $\hat{P}_{in}$ be the polynomial that interpolates $e_n(h)$ at all but one point of $E_n(h)$ and satisfies $\|\hat{P}_{in}\| = M_n(h)$. Then a brief argument (utilizing the fact that $E_n(f) = E_n(h)$) establishes that $\max_{-1 \leqslant x \leqslant 1} |\hat{P}_{in}(-x)| = M_n(f)$. Since $\max_{-1 \leqslant x \leqslant 1} |\hat{P}_{in}(-x)| = \max_{-1 \leqslant x \leqslant 1} |\hat{P}_{in}(x)|$, we have that

$$M_n(f) = M_n(h);$$

but $M_n[(-1)^{n+2} g_n] = M_n(h)$. Therefore $M_n(f)$ is again of precise order n. ∎

The next theorem is an immediate consequence of Theorem 6, (2.17), (3.4) and the remark following Theorem 5.

THEOREM 7. *Let $f(x) = e^x$, $x \in I$. Then*

(a) *$M_n(f)$ is of precise order n.*

(b) *If $v_1 < v_2 < \cdots < v_n$ are the zeros of the polynomial*

$$n(n^2-1)^{1/2}\, T_n(x) + (nx - 1)\, T_n'(x),$$

and if $x_1 < x_2 < \cdots < x_n$ are the interior extreme points of $e_n(f)$, then

$$0 < v_i - x_i < \frac{\beta}{n^3}$$

where β is independent of n.

Theorem 7 provides an estimate of the locations of the interior extreme points of the error function of $f(x) = e^x$. We note that historically the extreme points of either T_n or T_{n+1} have been used to estimate the location of the extreme points of the error curves $e_n(f)$ for functions satisfying $f^{(n+1)}(x) \neq 0$, $x \in I$ [14]. Theorem 7 provides a much tighter estimate to the location of the extreme points of the error curve for $f(x) = e^x$. A similar result is immediate for $e^{\alpha x}$, $\alpha \neq 0$.

A companion to Theorem 7 can be established for every $f \in \mathbf{F}$. In this latter setting the polynomial replacing that given in Theorem 7 is more complex (see (3.13)), and the distance between corresponding extreme points is $O(1/n^2)$.

4. Related Results

A number of observations of independent interest follow from the results established in Sections 2 and 3.

THEOREM 8. *Let $f \in \mathbf{F}$ with extreme set $-1 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Then*

$$\max_{-1 \leqslant x \leqslant 1} 2^n \prod_{j=0}^{n+1} |x - x_j| \tag{4.1}$$

is bounded.

Proof. Let $x^* \in (x_i, x_{i+1})$ be the value for which (4.1) is a maximum. Let q_{in} and Q_{n+1} be defined by (2.1) and (2.2), and let a_{n+1} be the coefficient of x^{n+1} in Q_{n+1}. Then as in (3.17),

$$(x^* - x_i)\, q_{in}(x^*) = (x^* - x_i)\, Q_{n+1}(x^*) - a_{n+1} \prod_{j=0}^{n+1} |x^* - x_j|.$$

This equation and (2.7) imply that

$$2^n \prod_{j=0}^{n+1} |x^* - x_j| \leqslant 2\, |x^* - x_i|\, M_n(f). \tag{4.2}$$

But (3.4) and Theorem 6 imply that the right-hand side of (4.2) is bounded independent of n. This observation completes the proof. ∎

We note (4.1) implies that

$$\max_{-1 \leqslant x \leqslant 1} \prod_{j=0}^{n+1} |x - x_j| = O\left(\frac{1}{2^{n+1}}\right). \tag{4.3}$$

It is known [3, p. 61] that

$$\min_{(x_0,\ldots,x_{n+1})} \max_{-1\leqslant x\leqslant 1} \prod_{j=0}^{n+1} |x-x_j| = \frac{1}{2^{n+1}}, \tag{4.4}$$

and that the $\{t_0, t_1,\ldots, t_{n+1}\}$ for which the minimum is attained are the $n+2$ zeros of T_{n+2}. Equation (4.3) suggests that the extreme set $E_n(f)$ of the error curve for any $f\in \mathbf{F}$ nearly produces a minimal monomial in the sense of (4.4)

THEOREM 9. *Let $f\in \mathbf{F}$, and let $E_n(f)=\{x_0,\ldots,x_{n+1}\}$ be the extreme set for $e_n(f)$. Define $P_{n+1}\in \prod_{n+1}$ by*

$$P_{n+1}(x_i)=f(x_i).$$

If $e_{n+1}(f)=f-B_{n+1}(f)$, then,

$$\frac{\|f-P_{n+1}\|}{\|e_{n+1}\|}\leqslant K,$$

where K is independent of n.

Proof. By the remainder theorem for interpolation

$$f(x)-P_{n+1}(x)=\frac{f^{(n+2)}(\xi)}{(n+2)!}\prod_{j=0}^{n+1}(x-x_j), \qquad -1<\xi<1.$$

Let x^* be a point for which $|f(x^*)-P_{n+1}(x^*)|=\|f-P_{n+1}\|$. Then by (2.15) and (3.3),

$$\frac{|f(x^*)-P_{n+1}(x^*)|}{\|e_{n+1}(f)\|}=\hat{\beta}\,2^{n+1}\prod_{j=0}^{n+1}|x^*-x_j|.$$

An application of Theorem 8 completes the proof. ∎

We note heuristically that Theorem 9 says interpolation at the extreme points of $e_n(f)$ is asymptotically as good as best approximating f by polynomials of degree at most $n+1$.

THEOREM 10. *Let $f\in \mathbf{F}$, and let $E_n(f)=\{x_0,\ldots,x_{n+1}\}$. If Q_{n+1} is defined by $E_n(f)$ as in (2.2), then*

$$\left\|\frac{e_n(f)}{\|e_n(f)\|}-Q_{n+1}\right\|\leqslant K\frac{\|e_{n+1}\|}{\|e_n\|}, \tag{4.5}$$

where K is independent of n.

Proof. We first note that

$$\frac{e_n(f)(x)}{\|e_n(f)\|} - Q_{n+1}(x) = \frac{f^{(n+2)}(\eta)}{\|e_n(f)\|\,(n+2)!} \prod_{j=0}^{n+1} (x - x_j), \qquad -1 < \eta < 1.$$

Again using (2.15), (3.3), and (4.1), we deduce (4.5). ∎

If $\|e_{n+1}\|/\|e_n\| \to 0$, then (4.5) implies that, asymptotically speaking, the behavior of Q_{n+1} resembles the behavior of T_{n+1}. We also observe that if $|f^{(n+2)}(\eta)/f^{(n+1)}(\xi)|$ is bounded, $-1 \leqslant \eta, \xi \leqslant 1$, independent of n, then (2.15) implies the right-hand side of (4.5) is $O(1/n)$.

5. Conclusion

In the present paper the precise order of the strong unicity constant $M_n(f)$ for any f from a particular class of functions $\mathbf{F}$ is shown to be n. Additionally, characteristics of the extremal sets $E_n(f)$ are examined.

The results of Sections 3 and 4 strongly suggest that if $E_n(f)$ contains precisely $n+2$ points, then the Lebesgue constant, λ_{n+1} [12, p. 89] determined by $E_n(f)$ is of precise order $\log n$ if ad only if $M_n(f)$ is of precise order n.

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